8. Brederlow, G., Witte, K.J. and Zinko, H., Investigation of the discharge structure in a noble gas alkali MHD generator plasma. AIAA Journal, Vol. 11, № 8, 1973.
9. Malikov, M.M., Experimental investigation on nonequilibrium plasma. Teplofizika Vysokikh Temperatur, Vol. 8, № 2, 1970.
10. Gilpin, R.R. and Zukoski, E.E., Experimental and theoretical studies of electrothermal waves. AIAA Journal, Vol. 7, № 8, 1969.
11. Sinkevich, O.A., Character of the loss of stability in a nonequilibrium magnetized plasma. PMM. Vol. 38, № 4, 1974.
12. Lagerstrom, P.A. and Casten, R.G., Basic concepts underlying singular perturbation techniques. AIAM Rev., Vol. 14, № 1, 1972.
13. Van Dyke, M., Perturbation Methods in Fluid Mechanics. Academic Press N.Y. and London, 1964.
14. Sinkevich, O.A., Waves of ionization and recombination in a weakly ionized magnetized plasma. PMM, Vol. 37, № 5, 1973.
15. Sinkevich, O.A., Ionization waves of finite amplitude in partly ionized plasma. Teplofizika Vysokikh Temperatur, Vol. 13, № 1, 1975.
16. Velikchov, E.P. and Dykhne, A.M., Plasma turbulence due to ionization instability in a strong magnetic field. Compt. Rendus 6-me Conférence Internat. Phénomènes ionisation gaz. Vol. 4, Paris, 1963.
17. Kulikovskii, A.G. and Liubimov, G.A., Magneto Hydrodynamics. Moscow, Fizmatgiz, 1962.
18. Stanyukovich, K.P., Unsteady Motion of Continuous Media, (English translation), Pergamon Press, Book № 04320, 1960.
19. Dykhne, A.M., Nonlinear effects in plasma in the presence of ionization instability. Proc. 5th Internat. Conf. Magnetohydrodynamic Electr. Power Generat., Vol. 2, Munich, 1971.

Translated by J. J. D.

UDC 533. 951. 7.

SOLUTION OF THE FIRST ORDER QUASI-LINEAR EQUATION THAT DEFINES THE EVOLUTION OF PLASMA TURBULENCE

PMM Vol. 40. №5. 1976. pp. 823-833

G. L. STENCHIKOV
(MOSCOW)

(Received January 16, 1976)

An asymptotic solution of the Cauchy problem is obtained for the first order quasi-linear equation. The field of characteristic curves is constructed. It is shown that for fairly considerable times the solution is discontinuous, but tends to a smooth stationary distribution. Numerical calculations obtained by the method of characteristics are presented. Results of the asymptotic and numerical analysis are in good agreement.

A homogeneous boundless plasma subjected to a strong high-frequency radiation [1] or a beam of particles [2] is unstable. Exponentially increasing with time t electro - magnetic oscillations with a wave vector $\mathbf{k}$ from some phase space region are induced in it during the initial linear stage. Nonlinear processes become substantial later when the energy spectral density $w(\mathbf{k}, t)$ reaches a high level. Such processes result in the saturation of the electromagnetic wave energy. The state of plasma under such conditions is called turbulent [3-5]. Investigation of the evolution of the energy spectral density of turbulent noise serves as the base for determining the form of functions of plasma particle distribution and the law governing the absorption and dispersion by the plasma of beams of particles or of powerful radiation. These questions are fundamental in the problem of the utilization of powerful light beams and of relativistic electrons for heating plasma.

1. The definition of evolution of plasma turbulence reduces in a number of cases [6-9] to solving the Cauchy problem.

$$\frac{\partial u}{\partial t} - \frac{1}{u} = u\left[1 - z^2 + \int_{z_1}^{z_2} dz' q(z'-z)\, u(z', t)\right] \tag{1.1}$$

$$u(z, 0) = u_0(z), \quad z_1 < z < z_2$$

where $u(z, t) \propto w(\mathbf{k}, t)$ is the angular distribution of pulsation energy. The variable z determines the deviation of oscillation propagation, which increases by increments $1 - z^2$, from the direction of the most intensive oscillation buildup. The integral term defines the nonlinear stabilizing mechanism — induced dispersion of waves on plasma particles used in [6-9]. The interaction between particles and electromagnetic waves results in the change of direction of the latter, and leads to a redistribution of the energy pulsations induced in the phase space region $|z| > 1$ The turbulent noise is transferred to the attenuation zone $z > 1$, where the increment is negative. The nonzero energy density is maintained in attenuation regions $|z| > 1$ owing to the equilibrium between energy absorption by plasma particles and the ther - mal excitation of pulsations (the term $1/u$ in the left part of (1.1). The "carrying out of noise" from the oscillation buildup region to that of attenuation stabilizes energy distribution in the phase space.

The position of maximum deviations z_1 and z_2, and the form of the core $q(\xi)$ depend on the particular physical system. All $q(\xi)$ cores are odd functions that tend to zero at rapid exponential rate with the characteristic scale $1/\beta \ll 1$ when $|\xi| \to \infty$ [6-9]. For functions $u(z, t)$, with the characteristic length of variation of the greater core width $1/\beta$, (1.1) is, for $\beta \to \infty$ equivalent to the quasi-linear first order equation [6, 8, 9].

$$\frac{\partial u}{\partial t} - \frac{1}{u} = u(1 - z^2) - \varepsilon u \frac{\partial u}{\partial z} \tag{1.2}$$

$$u(z, 0) = u_0(z), \quad -\infty < z < \infty$$

Investigation of the quasi-linear equation (1.2) is of interest for several reasons. First, it is possible to carry out an asymptotic analysis for $\varepsilon \to 0$. Second, the total information about the form of core q is contained in the single numerical parameter $\varepsilon \ll 1$. Third, it is possible to trace on a simple example how the convection effects — known in gas dynamics and nonlinear acoustics — are complicated by the peculiar to plasma phenomenon of oscillation buildup and thermal generation of oscillations.

Differential approximations for integro-differential equations similar to (1.1) were considered in [10, 12] without allowance for terms reflecting the buildup and thermal generation of noise. The stationary ordinary differential equation

$$\varepsilon u \frac{du}{dz} = \frac{1}{u} + u(1 - z^2) \tag{1.3}$$

which follows from (1.2) was used in [6, 8, 13] for defining the stationary energy distribution in the phase space.

2. Problem (1.2) is equivalent to the Cauchy problem for the characteristic system of ordinary differential equations, which after the transformation $y = u^2$ becomes

$$dy / dt = 2 + 2y(1 - z^2), \quad dz / dt = \varepsilon\sqrt{y} \tag{2.1}$$

$$z(0) = z_0, \quad y(0) = y_0(z_0) = u_0^2(z_0)$$

where the first equation defines the behavior of the unknown function $y(t)$ along the characteristic curve whose shape is determined by the second equation. Parameter ε is small ($\varepsilon \ll 1$).

The subsequent analysis is based on the method of joining asymptotic expansions [14, 15]. The pattern of behavior of characteristics of Eq. (1.2) in the (z, t)-plane appears in Fig. 1, where characteristic curves are shown by solid lines and the trajectories of shock discontinuities by dash lines. Roman numerals denote characteristics of various kinds which differ by the initial value of the coordinate $z_0 = z(0)$ as follows:

I: $z_0 < -1$; II: $z_0 = -1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\Delta < 0$, $0 < \alpha < {}^2/_5$; III: $z_0 = -1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\alpha > {}^2/_5$; IV: $z_0 = -1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\alpha = {}^2/_5$; V: $z_0 = -1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\Delta > 0$, $0 < \alpha < {}^2/_5$; VI: $-1 < z_0 < 1$; VII: $z_0 = 1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\Delta < 0$, $0 < \alpha < {}^2/_5$; VIII: $z_0 = 1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\alpha > {}^2/_5$; IX: $z_0 = 1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\alpha = {}^2/_5$; X: $z_0 = 1 + \Delta\varepsilon^\alpha$, $\Delta = O(1)$, $\Delta > 0$, $0 < \alpha < {}^2/_5$; XI: $z_0 > 1$

Arabic numerals denote regions through which pass the characteristic trajectories. In each of these regions the characteristic is defined by its asymptotic expansion.

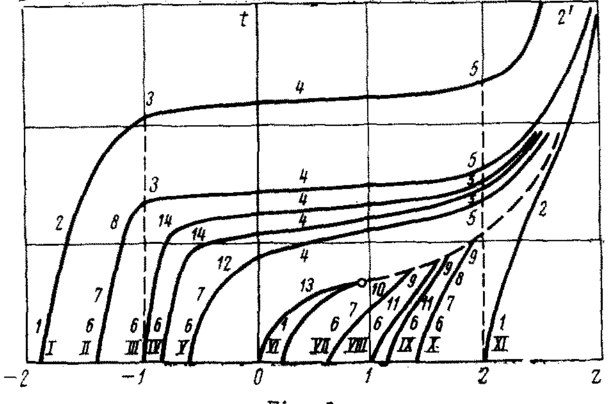

Fig. 1

Subsequently the analysis will be limited to the first terms of these expansions. Numbers of regions and subscripts of internal variables that define solutions in these regions coincide. Regions with identical expansions are denoted by identical numbers.

The complete description of solution requires the analysis of 46 zones along various characteristic curves. Below we present only the fundamental aspects of the asymptotic analysis, omitting details that can be obtained by analogy. The numbering of variables and of regions of nonhomogeneity coincides with that in Fig. 1.

We introduce variables

$$t_1 = t, \quad y_1 = y, \quad r_1 = (z - z_0) / \varepsilon \tag{2.2}$$

which transform (2.1) into system

$$dy_1 / dt_1 = 2 + 2y_1 (1 - z_0^2 - \varepsilon 2 r_1 z_0 - \varepsilon^2 r_1) \tag{2.3}$$

$$dr_1 / dt_1 = \sqrt{y_1}, \quad r_1(0) = 0, \quad y_1(0) = y_0 (z_0)$$

whose solution

$$y_1 = \left\{ y_0 (z_0) e^{-2(z_0^2-1)t_1} + \frac{1}{z_0^2 - 1} [1 - e^{-2(z_0^2-1)t_1}] \right\} \{1 + O(\varepsilon)\} \tag{2.4}$$

$$r_1 = (z_0^2 - 1)^{-1} \{ y_0^{1/2} (z_0) - y_1^{1/2} + \xi (t_1, z_0) \}$$

$$\xi (t_1, z_0) = \frac{1}{2} (z_0^2 - 1)^{-1/2} \ln \left[\frac{y_0^{1/2}(z_0) - (z_0^2 - 1)^{-1/2}}{y_0^{1/2}(z_0) + (z_0^2 - 1)^{-1/2}} \frac{y_1^{1/2} + (z_0^2 - 1)^{-1/2}}{y_1^{1/2} - (z_0^2 - 1)^{-1/2}} \right] \quad z_0^2 - 1 > 0$$

$$\xi (t_1, z_0) = (1 - z_0^2)^{-1/2} [\operatorname{arc\,tg} ((1 - z_0^2)\, y_0 (z_0))^{1/2} - \operatorname{arc\,tg} ((1 - z_0^2)\, y_1)^{1/2}] \quad z_0^2 - 1 < 0$$

is valid in region 1: $z - z_0 = O(\varepsilon)$, and $t = O(1)$ for all initial values of argument $z = z_0$ at the real axis, except the small neighborhoods of points $z = \pm 1$ (Fig. 1), where the instability increment $1 - z^2 = o(1)$ is small. This case should be considered separately. We extend the construction of characteristic curves of the kind I and XI (Fig. 1) with initial values $|z_0| > 1$ and introduce the substition

$$t_2 = \varepsilon t_1, \quad y_2 = y_1, \quad r_2 = \varepsilon r_1 = z - z_0 \tag{2.5}$$

which transforms Eq. (2.3) into system

$$\varepsilon dy_2 / dt_2 = 2 + 2y_2 (1 - z_0^2 - 2 r_2 z_0 - r_2^2) \tag{2.6}$$

$$dr_2 / dt_2 = \sqrt{y_2}$$

The integral (2.6) which satisfies the conditions of joining with solutions (2.4) in region 1

$$y_2 \to (z_0^2 - 1)^{-1}, \; r_2 \to t_2 (z_0^2 - 1)^{-1/2}; \; \varepsilon \to 0; \; y_1, \; t_1, \; r_1 = O(1)$$

is of the form

$$y_2 = (z^2 - 1)^{-1} \{1 + O(\varepsilon)\}; \; \ln |(z^2 - 1)^{1/2} - z| + z (z^2 - 1)^{1/2} = \tag{2.7}$$

$$\{2t_2 + \ln |(z_0^2 - 1)^{1/2} - z_0| + z_0 (z_0^2 - 1)^{1/2}\} \{1 + O(\varepsilon)\}$$

Formulas (2.7) define the stationary distribution of the energy spectral density, since the expression for the unknown function $y = (z^2 - 1)^{-1}$ does neither have an explicit dependence on time nor the dependence on initial conditions. In region 2 where $z - z_0 = O(1)$ and $t = O\,\varepsilon^{-1}$ and in regions which follow it along the characteristic the unknown function $u = \sqrt{y}$ conforms to the stationary equation (1.3) that was investigated in [13]. Equation of the characteristic curve is obtained from the second formula of (2. 1) and related conditions of joining. For $z_0 > 1$ expansion (2. 7) completes the construction of the characteristic of kind XI (Fig. 1).

Let us extend the construction of the trajectory of kind I with initial value $z_0 < -1$ (Fig. 1). Expansion (2. 7) joins in region 3, where $z + 1 = O(\varepsilon^{2/3})$, and

$$0 > t + (2\varepsilon)^{-1}\{\ln|(z_0^2 - 1)^{1/2} - z_0| + z_0(z_0^2 - 1)^{1/2}\} = O(\varepsilon^{2/3})$$

the solution

$$y_3 = W^2(r_3)\{1 + O(\varepsilon^{2/3})\} \tag{2.8}$$

$$t_3 = \left\{\int_a^\infty d\xi W^{-1}(\xi) + \int_\infty^{r_3} d\xi W^{-1}(\xi)\right\}\{1 + O(\varepsilon^{2/3})\}$$

$$t_3 = \{t_2 + 2^{-1}[\ln|(z_0^2 - 1)^{1/2} - z_0| + z_0(z_0^2 - 1)^{1/2}]\}\,\varepsilon^{-1/3},$$

$$y_3 = y_2\varepsilon^{2/3},\quad r_3 = (z + 1)\,\varepsilon^{-2/3}$$

which defines the rapid increase of the energy of noise.

Function $W(r_3)$ shown in Fig. 2 by the dash line satisfies the equation $dW/dr_3 = W^{-2} + 2r_3$ and has the following asymptotics [13]:

$$W = (-2r_3)^{-1/2},\quad r_3 \to -\infty;\quad W = r_3^2 + 1.633,\quad r_3 \to \infty$$

The conditions of joining principal terms of expansions do not provide means for determining the first integral in Eq. (2. 8) of the characteristic trajectory.

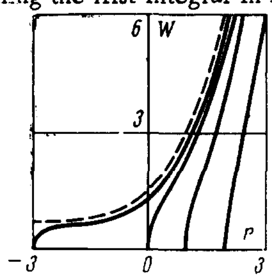

Fig. 2

However the dependence of the solution of system (2. 1) on the initial value z_0. is continuous. Using this it is possible to show that the indicated integral in (2. 8) is zero. A similar procedure is applied below in other cases without further explanations.

For $-1 < z < 2$ and $\{t + (2\varepsilon^{-1})[\ln|(z_0^2 - 1)^{1/2} - z_0| + z_0(z_0^2 - 1)^{1/2}]\} = O(1)$ the noise energy attains considerable values $y = O(\varepsilon^{-2})$ and the solution in this region is defined (Fig. 1) by formulas

$$y_4 = (z - z^3/3 + 2/3)^2\{1 + O(\varepsilon^{2/3})\} \tag{2.9}$$

$$(z + 1)^{-1} - 1/3\ln|z + 1| + 1/3\ln|z - 2| = -t_4\{1 + O(\varepsilon^{2/3})\}$$

$$y_4 = \varepsilon^{8/3}y_3,\quad t_4 = \varepsilon^{-2/3}t_3,\quad z = \varepsilon^{2/3}r_3 - 1$$

Expansion in the transition region when $z - 2 = O(\varepsilon)$ and

$$\{t + (2\varepsilon)^{-1}[\ln|(z_0^2 - 1)^{1/2} - z_0| + z_0(z_0^2 - 1)] + 1/3\ln\varepsilon\} = O(1)$$

(Fig. 1) is of the form

$$-1/3\,y_5^{1/2} + (6\sqrt{3})^{-1}\ln|[(3y_5)^{1/2} + 1]\,[(3y_5)^{1/2} - 1]^{-1}| = r_5\{1 + O(\varepsilon^{1/3})\} \tag{2.10}$$

$$\int_{-0,98}^{r_5} d\xi y_5^{-1/2}(\xi) = t_5\{1 + O(\varepsilon^{2/3})\};\quad r_5 = (z - 2)\varepsilon^{-1},$$

$$y_5 = y_4\varepsilon^{-2},\quad t_5 = t_4 + \frac{1}{3}\ln\varepsilon$$

It links (2.9) with the solution which defines the equilibrium between the absorption of plasma particle energy and the thermal noise generation

$$y_2 = (z^2 - 1)^{-1}\{1 + O(\varepsilon^{2/3})\}, \quad \ln|(z^2-1)^{1/2} - z| + z(z^2-1)^{1/2} = \tag{2.11}$$
$$\{2t_2' + \ln(2-3^{1/2}) + 2\cdot 3^{1/2}\}\{1 + O(\varepsilon^{2/3})\}$$

$$t_2' = \varepsilon t + 2^{-1}\{\ln|(z_0^2-1)^{1/2} - z_0| + z_0(z_0^2-1)^{1/2}\} + {}^1/_3\varepsilon\ln\varepsilon$$

The characteristic curves (2.11) are similar to the curves (2.7) already considered, but are shifted relative to the latter into the region of considerable times t (Fig. 1). Expansions (2.11) are valid in region

$$2': z > 2, t \gtrsim -(2\varepsilon)^{-1}\{\ln|(z_0^2-1)^{1/2} - z_0| + z_0(z_0^2-1)^{1/2}\} - {}^1/_3\ln\varepsilon + \varepsilon^{-1}.$$

3. For $z_0 = 1 + \Delta\varepsilon^\alpha$, $0 < \alpha < \infty$ and $\Delta = O(1)$ (characteristics of the kind VII — X in Fig. 1) the instability increment $1 - z_0^2 = O(\varepsilon^\alpha)$ is initially small. We introduce variables

$$t_6 = t_1, \quad y_6 = y_1, \quad r_6 = (z - 1 - \Delta\varepsilon^\alpha)\varepsilon^{-1} \tag{3.1}$$

which transfer (2.3) into the system

$$dy_6 / dt_6 = 2 + 2y_6(-2\Delta\varepsilon^\alpha - \Delta^2\varepsilon^{2\alpha} - 2\varepsilon^{1+\alpha} r_6\Delta - 2\varepsilon r_6 - \varepsilon^2 r_6^2) \tag{3.2}$$
$$dr_6 / dt_6 = \sqrt{y_6}$$

Its solution which satisfies initial conditions

$$r_6(0) = 0, \quad y_6(0) = y_0 \; (z_0 = 1 + \Delta\varepsilon^\alpha)$$

is of the form
$$y_6 = \{2t_6 + y_0(z_0)\}\{1 + O(\varepsilon^\alpha + \varepsilon)\} \tag{3.3}$$
$$r_6 = {}^1/_3\{[2t_6 + y_0(z_0)]^{3/2} - y_0^{3/2}(z_0)\}\{1 + O(\varepsilon^\alpha + \varepsilon)\}$$

which is valid in region 6: $z - z_0 = O(\varepsilon)$ and $t = O(1)$ (Fig. 1).

Further analysis is subdivided into several stages depending on which of the three conditions $0 < \alpha < {}^2/_5$, ${}^2/_5 < \alpha < \infty$ or $\alpha = {}^2/_5$ is valid.

Let us consider the first case $0 < \alpha < {}^2/_5$.

System (3.2) in variables

$$y_7 = \varepsilon^\alpha y_6, \quad t_7 = \varepsilon^\alpha t_6, \quad r_7 = \varepsilon^{3\alpha/2} r_6 \tag{3.4}$$

is of the form
$$dy_7 / dt_7 = 2 - 2y_7[2\Delta + \varepsilon^\alpha\Delta^2 + 2r_7\varepsilon^{1-5\alpha/2} + 2r_7\Delta\varepsilon^{1-3\alpha/2} + r_7^2\varepsilon^{2(1-2\alpha)}] \tag{3.5}$$
$$dr_7 / dt_7 = \sqrt{y_7}$$

The solution of these equations that satisfies the conditions of joining with (3.3)

$$y_7 \to 2t_7, \; 3r_7 \to (2t_7)^{3/2}; \; y_6, t_6, r_6 = O(1), \; \varepsilon \to 0$$

is defined by expansions

$$y_7 = (2\Delta)^{-1}\{1 - \exp[-4\Delta t_7]\}\{1 + O(\varepsilon^\alpha + \varepsilon^{1-5\alpha/2})\} \tag{3.6}$$
$$r_7 = -(2\Delta)^{-3/2}\{[1 - \exp(-4\Delta t_7)]^{1/2} + 2^{-1}\ln|[1 - \exp(-4\Delta t_7)]^{1/2} - 1| -$$
$$2^{-1}\ln|[1 - \exp(-4\Delta t_7)]^{1/2} + 1|\}\{1 + O(\varepsilon^\alpha + \varepsilon^{1-5\alpha/2})],$$
$$\Delta > 0$$
$$r_7 = (-2\Delta)^{-3/2}\{[\exp(-4\Delta t_7) - 1]^{1/2} - \operatorname{arctg}[\exp(-4\Delta t_7) - 1]^{1/2}\}\{1 + O(\varepsilon^\alpha + \varepsilon^{1-5\alpha/2})\}, \quad \Delta < 0$$

Region 7 is determined by formulas

$$(z - z_0) = O(\varepsilon^{3\alpha/2-1}), \quad t = O(\varepsilon^{-\alpha})$$

Let us extend the analysis to the case of $\Delta < 0$ which corresponds to the initial coordinate z_0 in the region $-1 < z < 1$, $z - 1 = O(\varepsilon^\alpha)$ of oscillation buildup. System (3.5) in variables

$$t_{10} = t_7 - (-2\Delta)^{-1}\ln\varepsilon^{5\alpha/2-1}, \quad y_{10} = \varepsilon^{2-5\alpha}y_7, \quad r_{10} = \varepsilon^{1-5\alpha/2}r_7 + \Delta \quad (3.7)$$

reduces to the equations

$$dy_{10} / dt_{10} = -4y_{10}r_{10} - \varepsilon^\alpha \, 2y_{10}r_{10}^2 + 2\varepsilon^{2-5\alpha} \quad (3.8)$$
$$dr_{10} / dt_{10} = \sqrt{y_{10}}$$

The asymptotic solution of this system which satisfies the conditions of joining (3.6)

$$y_{10} \to (-2\Delta)^{-1}\exp\{-4\Delta t_{10}\}, \quad r_{10} \to (-2\Delta)^{-3/2}\exp\{-2\Delta t_{10}\}$$
$$y_7, t_7, r_7 = O(1), \ \varepsilon \to 0$$

is represented by functions

$$t_{10} = (-2\Delta)^{-1}\{\ln|(r_{10} - \Delta)(r_{10} + \Delta)^{-1}| + {}^5/_2\ln(-2\Delta)\}\{1 + O(\varepsilon^\alpha + \varepsilon^{1-5\alpha/2})\}$$
$$y_{10} = (r_{10}^2 - \Delta^2)^2 \{1 + O(\varepsilon^\alpha + \varepsilon^{1-5\alpha/2})\} \quad (3.9)$$

These expansions are valid in region 10: $z - 1 = O(\varepsilon^\alpha)$, $t = O(\varepsilon^{-\alpha})$, with $0 < \alpha < {}^2/_5$. The characteristic trajectories (3.9) intersect each other. This property is independent of the initial distribution $y_0(z)$ and is the consequence of internal processes, viz. oscillation buildup in region $|z| < 1$ and transfer over the spectrum (the convection term). For fairly considerable times and arbitrary initial conditions the solution of problem (1.2) is discontinuous. The time t_s of discontinuity onset satisfies the inequality $t_s \leqslant \min_{0<\alpha<2/5} (-2\Delta\varepsilon^\alpha)^{-1}\ln\varepsilon^{5\alpha/2-1} = (-2\Delta)^{-1}\ln\varepsilon^{-1}$

The analysis of remaining variants shows that characteristics of the kind VIII — X do not intersect each other (Fig. 1). When $t \gtrsim \varepsilon^{-1}$ they coincide in the first approximation with the trajectory of the kind XI (2.7) that corresponds to parameter $z_0 = 1$.

Let the initial values of coordinates be $z_0 = -1 + \Delta\varepsilon^\alpha$, $\alpha > 0$, $\Delta = O(1)$ (characteristics if the kind II — V Fig. 1) Substituting variables

$$t_6 = t_1, \ y_6 = y_1, \ r_6 = (z + 1 - \Delta\varepsilon^\alpha)\varepsilon^{-1} \quad (3.10)$$

we obtain from (2.3) a system of equations and its integrals which satisfy initial conditions

$$dy_6 / dt_6 = 2 + 2y_6(\varepsilon^\alpha 2\Delta - \varepsilon^{2\alpha}\Delta^2 - \varepsilon^{1+\alpha}2r_6\Delta + \varepsilon 2r_6 - \varepsilon^2 r_6^2) \quad (3.11)$$
$$dr_6 / dt_6 = \sqrt{y_6}$$
$$y_6 = \{2t_6 + y_0(z_0)\} \{1 + O(\varepsilon^\alpha + \varepsilon)\} \quad (3.12)$$
$$r_6 = {}^1/_3 \{[2t_6 + y_0(z_0)]^{3/2} - y_0^{3/2}(z_0)\} \{1 + O(\varepsilon^\alpha + \varepsilon)\}$$

In the first approximation (3.11) and (3.12) are the same as (3.2) and (3.3). The analysis is subdivided into several stages, viz. $0 < \alpha < {}^2/_5$, $\alpha > {}^2/_5$, and $\alpha = {}^2/_5$. Let us consider the second of these, namely $\alpha > {}^2/_5$. The substitution

$$t_{14} = \varepsilon^{2/5}t_6, \quad r_{14} = \varepsilon^{3/5}r_6, \quad y_{14} = \varepsilon^{2/5}y_6 \quad (3.13)$$

reduces system (3.11) to

$$dy_{14} / dt_{14} = 2 + \varepsilon^{-2/5}2y_{14}(2\Delta\varepsilon^\alpha - \varepsilon^{2\alpha}\Delta^2 - \varepsilon^{2/5+\alpha}2r_{14}\Delta + \varepsilon^{2/5}2r_{14} - \varepsilon^{4/5}r_{14}^2), \quad dr_{14} / dt_{14} = \sqrt{y_{14}} \quad (3.14)$$

Substituting $W = y_{14}^{1/2}$ and using the equality $dy_{14}/dt_{14} = y_{14}^{1/2} dy_{14}/dr_{14}$, we obtain equations for the principal terms of the expansions of unknown functions

$$dW / dr_{14} = W^{-2} + 2r_{14}, \quad dr_{14} / dt_{14} = W \tag{3.15}$$

The conditions of joining

$$W \to (3r_{14})^{1/3}, \; r_{14} \to {}^{1}/_{3}\,(2t_{14})^{3/2};\; r_6,\, t_6,\, y_6 = O\,(1),\; \varepsilon \to 0 \tag{3.16}$$

follow from (3.12). Integral curves of the first of Eqs. (3.15) are shown in Fig. 2. Conditions (3.16) are satisfied by the trajectory which passes through the coordinate origin $W = W_0\,(r_{14})$ The solution

$$y_{14} = W_0^2\,(r_{14}), \quad t_{14} = \int_0^{r_{14}} d\xi W_0^{-1}\,(\xi) \tag{3.17}$$

shows, as (2.8), the rapid increase of noise energy due to the increment which is linear with respect to the coordinate. Note that (2.8) and (3.17) satisfy the same system of Eqs. (3.15) but for different initial conditions. The dimensions of region 14 are the same as those of region 3, but the former is displaced in the direction of shorter times t: $z + 1 = O\,(\varepsilon^{2/5})$, $t = O\,(\varepsilon^{-2/5})$. Functions (3.17) merge with solution of the kind (2.9) in region $-1 < z < 2$ Fig. 1 .

Further analysis can be carried out on the basis of the stationary equation (1.3) [13]. Investigation of all possible variants shows that the characteristic trajectories of the kind II—V with the initial condition $z_0 = -1 + \Delta\varepsilon^{\alpha}$, $\alpha > 0$ and $\Delta = O\,(1)$ do not intersect each other, and for $t \geqslant {}^{1}/_{3}\ln \varepsilon^{-1} + \varepsilon^{-1}$ coincide in the first approximation with the characteristic of the XI kind which corresponds to the initial coordinate $z_0 = 2$,(see (2.7)).

4. The unknown function increases exponentially in accordance with (2.4) along characteristic curves with parameter z_0 within the range $-1 < z_0 < 1$. When $t = O\,[(1 - z_0^2)^{-1}\ln\varepsilon^{-1}]$ the noise energy attains considerable values $y = O\,(\varepsilon^2)$. The substitution of variables $z = \varepsilon r_1 + z_0$, $y_{13} = \varepsilon^2 y_1$ and $t_{13} = t_1 - (1 - z_0^2)^{-1}\ln\varepsilon^{-1}$ transforms system (2.3) to one of the form

$$dy_{13} / dt_{13} = 2\varepsilon^2 + 2y_{13}\,(1 - z^2)$$
$$dz / dt_{13} = \sqrt{y_{13}}$$

With the use of equality $dy_{13} / dt_{13} = y_{13}^{1/2} dy_{13} / dr_{13}$, for the principal terms of expansion we obtain the equation

$$\frac{d}{dz}\sqrt{y_{13}} = 1 - z^2, \quad \frac{dz}{dt_{13}} = \sqrt{y_{13}} \tag{4.1}$$

The nonstationary "turbulent" solution

$$\begin{gathered} y_{13} \to [(1 - z_0^2)(z - z_0)]^2, \; (z - z_0) \to (1 - z_0^2)^{-1}\,\{y_0\,(z_0) + (1 - z_0^2)^{-1}\}^{1/2} \exp\,[(1 - z_0^2)t_{13}] \\ r_{13}\; y_1,\; t_1 = O\,(1),\; \varepsilon \to 0 \end{gathered} \tag{4.2}$$

which satisfies the conditions of joining (2.4), is defined by expansions

$$\begin{gathered} y_{13} = (z - {}^{1}/_{3}z^3 - z_0 + {}^{1}/_{3}z_0^3)^2\,\{1 + O\,(\varepsilon)\} - \\ 3\,[A\ln\,(z - z_0) + B\ln\,(z - Z_1) + C\ln\,(Z_2 - z)] + C_{13} = t_{13}\,\{1 + O\,\varepsilon)\} \end{gathered} \tag{4.3}$$

$$Z_1 = -z_0/2-(3-{}^3/_4 z_0^2)^{1/2},\ Z_2 = -z_0/2+(3-{}^3/_4 z_0^2)^{1/2} \tag{4.3}$$
$$A = {}^1/_3(z_0^2-1)^{-1},\ B = {}^1/_3[z_0(3-{}^3/_4 z_0^2)^{1/2}+2(1-z_0^2/4)]^{-1}$$
$$C = {}^1/_3[2(1-z_0^2/4)-z_0(3-{}^3/_4 z_0^2)^{1/2}]^{-1}$$
$$C_{13} = 3B\ln(z_0-Z_1)+3C\ln(Z_2-z_0)+$$
$$(1-z_0^2)^{-1}\ln\{(1-z_0^2)[y_0(z_0)+(1-z_0^2)^{-1}]^{-1/2}\}$$

Region 13 is determined by formulas

$$z-z_0 = O(1),\quad t+(1-z_0^2)^{-1}\ln\varepsilon = O(1)$$

The inequality $y_{13}(z, z_0') > y_{13}(z, z_0)$ is satisfied for fixed z along characteristics of the kind VI and various initial values of the

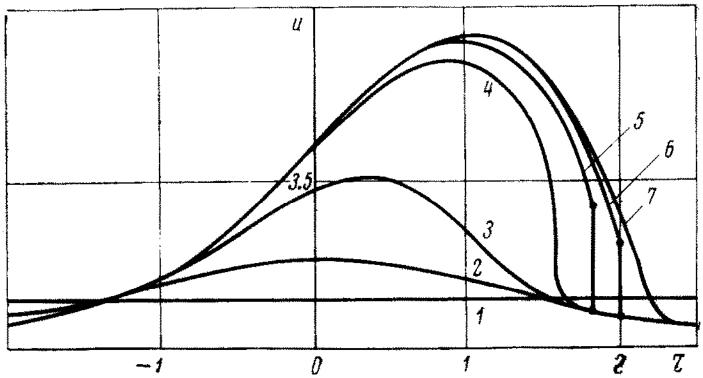

Fig. 3

coordinate $z_0' < z_0$. Hence the characteristic trajectories (4.3) with lower values of the initial coordinate z_0 approach the trajectories with higher values of that coordinate and intersect the latter. Such pattern of the field of characteristic curves (Fig. 1) corresponds to a discontinuous solution of the quasi-linear equation (1.2).

Let us consider trajectories with small initial values of coordinate $|z_0| \ll 1$. The relationship

$$t_{13} = f(z, z_0) \approx \frac{1}{2}\ln\frac{3z^2}{3-z^2} - \frac{1}{2}\ln[1+y_0(z_0)] \tag{4.4}$$
$$+ z_0\left[\frac{3(z^2-2)}{2z(3-z^2)} + \frac{\sqrt{3}}{4}\ln\frac{\sqrt{3}+z}{\sqrt{3}-z}\right]$$

follows from (4.3).

The point of intersection of two characteristics close to each other is defined by the equation

$$\frac{\partial f}{\partial z_0} \approx \frac{3(z^2-2)}{2z(3-z^2)} + \frac{\sqrt{3}}{4}\ln\frac{\sqrt{3}+z}{\sqrt{3}-z} - \frac{1}{2}[1+y_0(z_0)]^{-1}\frac{dy_0(z_0)}{dz_0} = 0 \tag{4.5}$$

which makes it possible to predict the point of incipient discontinuity (z_s, t_s). It follows from (4.5) that when $y_0(z) \equiv 1$ then $z_s \simeq \sqrt{2}$, and $t_s \approx \ln(\sqrt{3}/\varepsilon)$.

Equation (1.2) with initial condition $u_0(z) \equiv 1$ and parameter $\varepsilon = 0.3$ was solved numerically by the method of characteristics [16]. In Figs. 3-5 are plotted solutions (curves denoted by numerals 1-7 relate to times $t = 0, 0.4, 1.0, 1.8, 2.2, 2.6, 3.6$), the field of characteristics in the (z, t)- plane, and the amplitude evolution at the time of shock discontinuity $[u]$. Asymptotic estimates of the time of shock wave onset and attenuation $t_s \approx \ln(\sqrt{3}/\varepsilon) = 1.75$ and $t_\infty \approx {}^1/_3 \ln \varepsilon^{-1} + \varepsilon^{-1} \approx 3.7$ is in agreement with the results of numerical computations (Fig. 5).

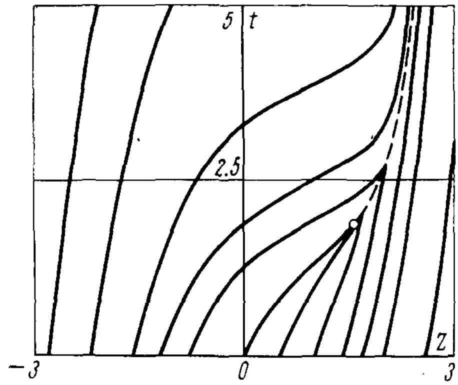

Fig. 4

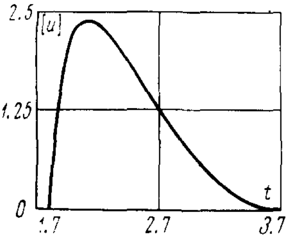

Fig. 5

The dincontinuous general solution of the correctly formulated Cauchy problem for the quasi-linear equation must satisfy a certain condition at the discontinuity trajectory $z = z_s(t)$. That condition may be obtained by using the divergent form of Eq. (1.2). The correct condition at the discontinuity satisfies the inequality which in the notation used here is of the form [17, 18]

$$\varepsilon u(z_s + 0) \leqslant dz_s / dt \leqslant \varepsilon u(z_s - 0) \tag{4.6}$$

In gasdynamics this condition corresponds to the law of entropy increase and excludes solutions of the kind of rarefaction shock waves. Equation (1.2) corresponds to a continuum of correct conditions at the discontinuity [17, 18]. On physical considerations we select the condition

$$dz_s / dt = \varepsilon / 2 \, [u(z_s + 0) + u(z_s - 0)] \tag{4.7}$$

which corresponds to the conservation of the noise energy stream through the shock front in the wave vector space.

The solution of problem (1.2) is discontinuously independent of the initial distribution $y_0(z)$, although its detailed form, for instance, the number of simultaneously existing discontinuities is dependent on the initial condition. Characteristics whose parameter z_0 is outside the interval $-1 < z_0 < 2$ do not intersect trajectories of the shock discontinuity, while those with that parameter within the range $-1 < z_0 < 2$ form a shock wave. Characteristic trajectories with initial conditions from the neighborhood of point $z = -1$ asymptotically approach for $t \gtrsim {}^1/_3 \ln \varepsilon^{-1} + \varepsilon^{-1}$ the curve emerging from point $t = 0, z = 2$. Hence the amplitude of the shock discontinuity $[u] = |u(z_s + 0) - u(z_s - 0)|$, which is proportional the the difference of the tangents of inclination angles of intersecting characteristics at the intersection point, tends to zero when $t > {}^1/_3 \ln \varepsilon^{-1} + \varepsilon^{-1}$. After this the stationary distribution, which is the same as that obtained in [13], is established. By virtue of (4.6) this qualitative result is valid for any correct conditions at the discontinuity. Conclusions reached by asymptotic analysis are confirmed by numerical computations (Figs. 1, 3-5).

The evolution of plasma turbulence saturated by spectral transfer occurs in three time stages (Fig. 3). First, the plasma noise increases exponentially in the interval $0 < t < \ln\varepsilon^{-1}$ in the region $|z| < 1$ of oscillation buildup and, then, decreases in the attenuation region $|z| > 1$.

In the interval $\ln\varepsilon^{-1} < t < \varepsilon^{-1}$ the noise energy attains considerable values $u = O(\varepsilon^{-1})$, and the intensive spectral transfer results in the onset of a shock wave in the phase space. In the case of a discontinuous solution Eqs. (1.1) and (1.2) are not asymptotically equivalent. Numerical computations of problem (1.1) in [7] show that the shock wave has a dispersion structure. For $t \gtrsim {}^1/_3 \ln\varepsilon^{-1}$ the solution of the integro-differential equation (1.1) has an oscillatory character with a scale of $1/\beta$ equal to the core width.

Further behavior of the spectral density of oscillation energy is determined by the attenuation of the shock wave at $t \gtrsim \ln\varepsilon^{-1} + \varepsilon^{-1}$ and the establishment of a smooth stationary distribution.

The author thanks V. V. Aleksandrov, V. V. Pustovalov, V. P. Silin, and V. T. Tikhonchuk for their help in formulating the problem, its discussion, and interest in it.

REFERENCES

1. Silin, V. P., Parametric Effect of High Power Radiation on Plasma. Moscow, "Nauka", 1973.
2. Mikhailovskii, A. B., Theory of Plasma Instabilities. Vol. 1, Moscow, Atomizdat, 1970.
3. Kadomtsev, B. B., Plasma turbulence. Collection: Problems of Plasma Theory, № 4, Moscow, Atomizdat, 1964.
4. Tsytovich, V. N., Theory of Turbulent Plasma. Moscow, Atomizdat, 1971.
5. Pustovalov, V. V. and Silin, V. P., Nonlinear theory of wave interaction in plasma. Collection: Tr. Fizichesk. Inst. Akad. Nauk SSSR, Vol. 61, 1972.
6. Pustovalov, V. V., Silin, V. P. and Tikhonchuk, V. T., Nonlinear transformation of radiation into plasma waves. ZhETF, Vol. 65, № 5, 1973.
7. Andreev, N. E., Pustovalov, V. V., Silin, V. P. and Tikhonchuk, V. T., Nonstationary parametric plasma turbulence. Letters to ZhETF, Vol. 18, № 10, 1973.
8. Andreev, N. E., Pustovalov, V. V., Silin, V. P. and Tikhonchuk, V. T., Relaxation of parametric plasma turbulence. Kvantovaia Elektronika, Vol. 1, № 5, 1974.
9. Golikov, Iu. V., Pustovalov, V. V., Silin, V P. and Tikhonchuk, V. T., Nonlinear theory of noise generation by electron stream in magnetically active plasma. Kratkie Soobshcheniia po Fizike, № 3, 1974.
10. Kompaneets, A. S., On the establishment of thermal equilibrium between quanta and electrons. ZhETF, Vol. 31, № 5, 1956.
11. Galeev, A. A., Kerpman, V. I. and Sagdeev, P. Z., On a particular solvable problem of the theory of turbulent plasma. Dokl. Akad. Nauk SSSR, Vol. 157, № 5, 1964.
12. Zel'dovich, Ia. B. and Siuniaev, R. A., Shockwave structure in the radiation spectrum at Bose-condensation of photons. ZhETF, Vol. 62, № 1, 1972.
13. Aleksandrov, V. V., Pustovalov, V. V., Silin, V. P., Stenchikov, G. L. and Tikhonchuk, V. T., On stationary spectral energy density

of turbulent plasma noise. Izv. VUZ, Radiofizika, Vol. 17, № 10, 1974.
14. Van Dyke, M., Perturbation Methods in Fluid Mechanics. Academic Press, New York and London, 1964.
15. Cole, J., Perturbation Methods in Applied Mathematics. Blaisdell Waltham, Mass., U. S. A., 1968.
16. Godunov, S. K. and Riaben'kii, V. S., Difference Schemes. Moscow, "Nauka", 1973.
17. Courant, R., Partial Differential Equations. New York, Interscience, 1962.
18. Oleinik, O. A., Discontinuous solutions of nonlinear differential equations. Uspekhi Matem. Nauk, Vol. 12, № 3, 1957.

Translated by J. J. D.

UDC 533.7.

ON PROPER TRANSONIC FLOWS OF RADIATIVE GAS IN CHANNELS WITH SLIGHTLY VARYING PARAMETERS

PMM Vol. 40, №5, 1976, pp. 834-845

V. V. ALEKSANDROV and A. A. FROLOVA
(Moscow)

(Received February 26, 1975)

Flows of gas at velocities close to the isentropic and isothermal speeds of sound in channels with slowly changing temperature and curved walls are considered. The model takes into account the convection nonlinearity resulting from the cumulative effect of perturbation propagation. It also permits the analysis of the arbitrary effect of radiation on the motion of gas. The derived nonlinear system of equations defines the flow of gas in channels whose transverse optical thickness is of the order of unity. Similar equations for quasi-isentropic flows appear in [1].

1. Input equations. Let us consider the stationary equilibrium flow of an inviscid non-heat-conducting radiative gas in a channel with plane or axial symmetry. The channel walls are assumed to be nearly parallel planes or, in the case of axial symmtry, to have a nearly cylindrical surface. In the plane case the channels are assumed to be symmetric about the plane $y = 0$.

We assume that the input of radiation to the internal energy density and to the pressure is small. The motion of such medium is defined by the equations

$$\frac{\partial \rho u}{\partial x} + \frac{1}{y^{d-1}} \frac{\partial}{\partial y} y^{d-1} \rho v = 0 \tag{1.1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0$$

$$\rho T \left(u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} \right) + \operatorname{div} \mathbf{q} = 0$$

where p is the pressure, ρ is the density, T is the temperature, s is the entropy of